Numerical solutions of ordinary differential equations

Frames 1 to 69

Learning outcomes

When you have completed this Programme you will be able to:

- Derive a form of Taylor's series from Maclaurin's series and from it describe a function increment as a series of first and higher-order derivatives of the function
- Describe and apply by means of a spreadsheet the Euler method, the Euler–Cauchy method and the Runge–Kutta method for first-order differential equations
- Describe and apply by means of a spreadsheet the Euler second-order method and the Runge–Kutta method for second-order ordinary differential equations
- Describe and apply by means of a spreadsheet a simple predictor–corrector method.

Prerequisite: Engineering Mathematics (Fifth Edition)
Programme F.4 (Uses of a spreadsheet)

Introduction

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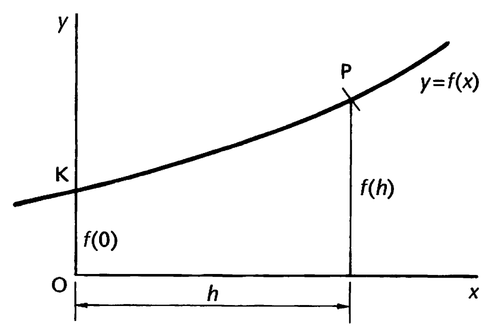
The range of differential equations that can be solved by straightforward analytical methods is relatively restricted. Even solution in series may not always be satisfactory, either because of the slow convergence of the resulting series or because of the involved manipulation in repeated stages of differentiation.

In such cases, where a differential equation and known boundary conditions are given, an approximate solution is often obtainable by the application of numerical methods, where a numerical solution is obtained at discrete values of the independent variable.

The solution of differential equations by numerical methods is a wide subject. The present Programme introduces some of the simpler methods, which nevertheless are of practical use.

Taylor's series

Let us start off by briefly revising the fundamentals of Maclaurin's and Taylor's series.



Maclaurin's series for $f(x)$ is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^n}{n!}f^n(0) + \ldots \qquad (1)$$

and expresses the function $f(x)$ in terms of its successive derivatives at $x = 0$, i.e. at the point K.

Therefore, at P, $\quad f(h) = \ldots\ldots\ldots\ldots$

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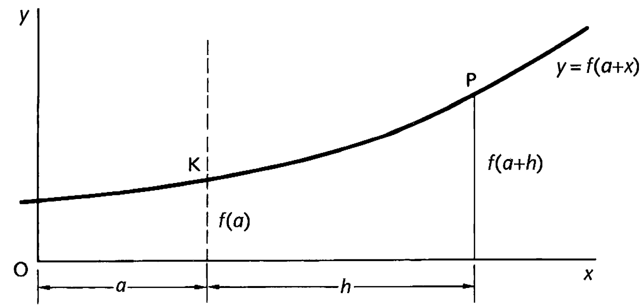
$$\boxed{f(h) = f(0) + hf'(0) + \frac{h^2}{2!}f''(0) + \ldots + \frac{h^n}{n!}f^n(0) + \ldots} \quad (2)$$



If the y-axis and origin are moved a units to the left, the equation of the same curve relative to the new axes becomes $y = f(a + x)$ and the function value at K is $f(a)$.

At P, $f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^n}{n!}f^n(a) + \ldots$

This is one common form of Taylor's series.

Make a note of it and then move on

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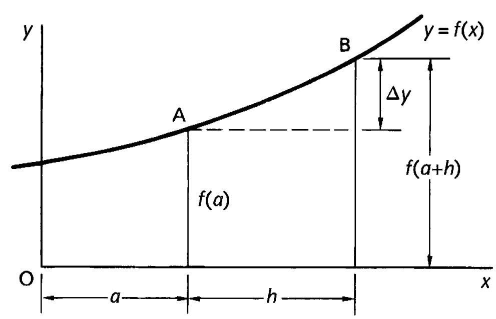
Function increment



If we know the function value $f(a)$ at A, i.e. at $x = a$, we can apply Taylor's series to determine the function value at a neighbouring point B, i.e. at $x = a + h$.

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \ldots \quad (3)$$

The *function increment* from A to B $= \Delta y = f(a + h) - f(a)$

i.e. $\quad f(a + h) = f(a) + \Delta y$

where $\Delta y = hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \ldots$

This entails evaluation of an infinite number of derivatives at $x = a$: in practice an approximation is accepted by restricting the number of terms that are used in the series.

▶

This approximation of Taylor's series forms the basis of several numerical methods, some of which we shall now introduce. It should be noted that these early examples have been selected because exact solutions can also be found. The purpose of this is to enable a comparison between the results obtained by a particular method with those obtained from an exact solution, and so to demonstrate the accuracy of the method.

On then to the next frame

First-order differential equations

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Numerical solution of $\dfrac{dy}{dx} = f(x, y)$ with the initial condition that, at $x = x_0$, $y = y_0$.

Euler's method

The simplest of the numerical methods for solving first-order differential equations is *Euler's method*, in which the Taylor's series

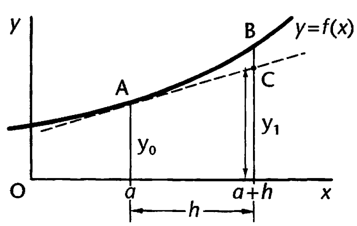
$$f(a+h) = f(a) + hf'(a) \boxed{+\frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \dots}$$

is truncated after the second term to give

$$f(a+h) \approx f(a) + hf'(a) \tag{4}$$

This is a severe approximation, but in practice the 'approximately equals' sign is replaced by the normal 'equals' sign, in the knowledge that the result we obtain will necessarily differ to some extent from the function value we seek. With this in mind, we write

$$f(a+h) = f(a) + hf'(a)$$



If h is the interval between two near ordinates and if we denote $f(a)$ by y_0, then the relationship

$$f(a+h) = f(a) + hf'(a)$$

becomes

$$y_1 = y_0 + h(y')_0 \tag{5}$$

Hence, knowing y_0, h and $(y')_0$, we can compute y_1, an approximate value for the function value at B.

Make a note of result (5): we shall be using it quite a lot.

Then move on for an example

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Example 1

Given that $\dfrac{dy}{dx} = 2(1 + x) - y$ with the initial condition that at $x = 2$, $y = 5$, we can find an approximate value of y at $x = 2{\cdot}2$, as follows.

We have $\quad y' = 2(1 + x) - y \quad$ with $\quad x_0 = 2,\ y_0 = 5$

$$\therefore\ (y')_0 = \dots\dots\dots\dots$$

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$$\boxed{(y')_0 = 1}$$

We obtain this by substituting x_0 and y_0 in the given equation:

$$(y')_0 = 2(1 + x_0) - y_0 = 2(1 + 2) - 5 \quad \therefore\ (y')_0 = 1$$

So we have $x_0 = 2$; $y_0 = 5$; $(y')_0 = 1$; $x_1 = 2{\cdot}2$; $h = 0{\cdot}2$.

By Euler's relationship:

$$y_1 = y_0 + h(y')_0 \quad \therefore\ y_1 = \dots\dots\dots\dots$$

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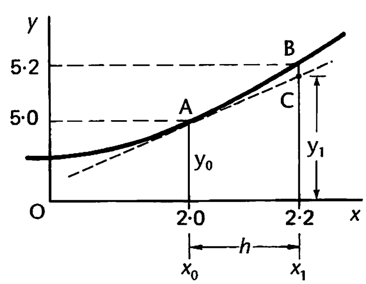
$$\boxed{y_1 = 5{\cdot}2}$$

Because

$$y_1 = y_0 + h(y')_0 = 5 + (0{\cdot}2)1 = 5{\cdot}2$$



At B, $x_1 = 2{\cdot}2$; $\ y_1 = 5{\cdot}2$; and

$$(y')_1 = \dots\dots\dots\dots$$

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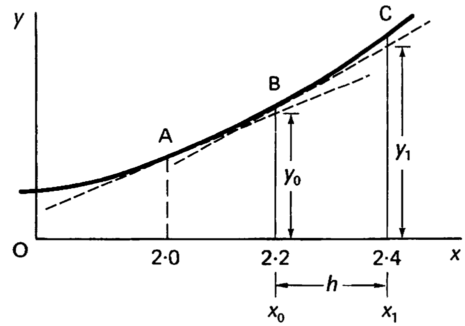
$$\boxed{(y')_1 = 1{\cdot}2}$$

$$(y')_1 = 2(1 + x_1) - y_1 = 2(1 + 2{\cdot}2) - 5{\cdot}2 = 1{\cdot}2$$



If we take the values of x, y and y' that we have just found for the point B and treat these as new starter values x_0, y_0, $(y')_0$, we can repeat the process and find values corresponding to the point C.

At B, $x_0 = 2{\cdot}2$; $y_0 = 5{\cdot}2$; $(y')_0 = 1{\cdot}2$; $x_1 = 2{\cdot}4$.

Then at C: $y_1 = \ldots\ldots\ldots\ldots$; $(y')_1 = \ldots\ldots\ldots\ldots$

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$$\boxed{y_1 = 5{\cdot}44; \quad (y')_1 = 1{\cdot}36}$$

$$y_1 = y_0 + h(y')_0 = 5{\cdot}2 + (0{\cdot}2)1{\cdot}2 = 5{\cdot}44$$
$$(y')_1 = 2(1 + x_1) - y_1 = 2(1 + 2{\cdot}4) - 5{\cdot}44 = 1{\cdot}36$$

So we could continue in a step-by-step method. At each stage, the determined values of x_1, y_1 and $(y')_1$ become the new starter values x_0, y_0 and $(y')_0$ for the next stage.

Our results so far can be tabulated thus

x_0	y_0	$(y')_0$	x_1	y_1	$(y')_1$
2·0	5·0	1·0	2·2	5·2	1·2
2·2	5·2	1·2	2·4	5·44	1·36
2·4	5·44	1·36			

Continue the table with a constant interval of $h = 0{\cdot}2$. The third row can be completed to give

$x_1 = \ldots\ldots\ldots\ldots$; $y_1 = \ldots\ldots\ldots\ldots$; $(y')_1 = \ldots\ldots\ldots\ldots$

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$$x_1 = 2{\cdot}6; \quad y_1 = 5{\cdot}712; \quad (y')_1 = 1{\cdot}488$$

Because

$$x_1 = x_0 + h = 2{\cdot}4 + 0{\cdot}2 = 2{\cdot}6$$
$$y_1 = y_0 + h(y')_0 = 5{\cdot}44 + (0{\cdot}2)1{\cdot}36 = 5{\cdot}712$$
$$(y')_1 = 2(1 + x_1) - y_1 = 2(1 + 2{\cdot}6) - 5{\cdot}712 = 1{\cdot}488$$

Now you can continue in the same way and complete the table for

$$x = 2{\cdot}0,\ 2{\cdot}2,\ 2{\cdot}4,\ 2{\cdot}6,\ 2{\cdot}8,\ 3{\cdot}0$$

Finish it off and compare results with the next frame

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Here is the result.

x_0	y_0	$(y')_0$	x_1	y_1	$(y')_1$
2·0	5·0	1·0	2·2	5·2	1·2
2·2	5·2	1·2	2·4	5·44	1·36
2·4	5·44	1·36	2·6	5·712	1·488
2·6	5·712	1·488	2·8	6·009 6	1·590 4
2·8	6·009 6	1·590 4	3·0	6·327 68	1·672 32
3·0	6·327 68	1·672 32			

In practice, we do not, in fact, enter the values in the right-hand half of the table, but write them in directly as new starter values in the left-hand section of the table.

x_0	y_0	$(y')_0$
2·0	5·0	1·0
2·2	5·2	1·2
2·4	5·44	1·36
2·6	5·712	1·488
2·8	6·009 6	1·590 4
3·0	6·327 68	1·672 32

The particular solution is given by the values of y against x and a graph of the function can be drawn.

Draw the graph of the function carefully on graph paper.

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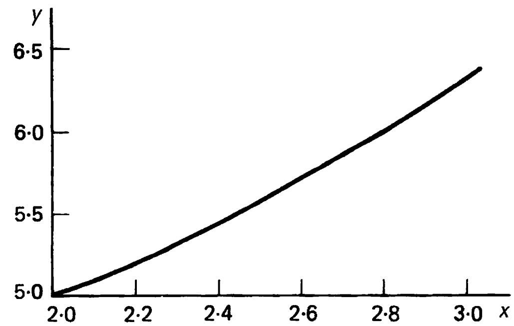
Graph of the solution of $\frac{dy}{dx} = 2(1+x) - y$ with $y = 5$ at $x = 2$.



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It is an advantage to plot the points step-by-step as the results are built up. In that way, one can check that there is a smooth progression and that no apparent errors in the calculations occur at any one stage.

The differential equation $\frac{dy}{dx} = 2(1+x) - y$ can be solved by the integration factor method (see *Engineering Mathematics, Fifth Edition*, Programme 24) to give the solution

$$y = 2x + e^{2-x}$$

and in the following table we compare our results with the actual values to determine the errors.

x	y (Euler)	y (actual)	Absolute error
2·0	5·0	5·0	0
2·2	5·2	5·218 731	0·018 731
2·4	5·44	5·470 320	0·030 320
2·6	5·712	5·748 812	0·036 812
2·8	6·009 6	6·049 329	0·039 729
3·0	6·327 68	6·367 879	0·040 199

The errors involved in the process are shown. These errors are due mainly to ....

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the fact that Taylor's series was truncated after the second term

By now you will appreciate the amount of arithmetic manipulation involved in solving these differential equations – a large amount of which is repetitive. To avoid the tedium and to make the computations more efficient we shall resort to the use of a spreadsheet. If the use of a spreadsheet is a totally new experience to you then you are referred to Programme F.4 of *Engineering Mathematics, Fifth Edition*, where the spreadsheet is introduced as a tool for constructing graphs of functions. If you have a limited knowledge then you will be able to follow the text from here. The spreadsheet we shall be using here is Microsoft Excel, though all commercial spreadsheets possess the equivalent functionality. Alternatively, an iteration process can be used in any computer algebra package such as *Derive*, *Maple* or *Mathematica*.

Open your spreadsheet and in cell A1 enter the letter n and press **Enter**. In this first column we are going to enter the iteration numbers. In cell A2 enter the number 0 and press **Enter**. Place the cell highlight in cell A2 and highlight the block of cells A2 to A12 by holding down the mouse button and wiping the highlight down to cell A12. Click the **Edit** command on the Command bar and point at **Fill** from the drop-down menu. Select **Series** from the next drop-down menu and accept the default **Step value** of 1 by clicking **OK** in the Series window.

The cells A3 to A12 fill with ....

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The numbers 1 to 10

In cell B1 enter the letter x – this column is going to contain the successive x-values for which the y-value is going to be enumerated. In cell B2 enter the number 2 – the initial x-value. We now could fill the column in much the same way as we filled the first column, but we have a better way.

Place the cell highlight in cell F1 and enter the number 0·2 – this is the value of h, the increment in x. Now place the cell highlight in cell B3 and enter the formula

=B2+$F$1 followed by **Enter** (uppercase or lowercase, it does not matter)

The number 2·2 appears in cell B3. Place the cell highlight in cell B3, click the **Edit** command and select **Copy** from the drop-down menu. You have now copied the contents of cell B3 to the clipboard. Now place the cell highlight in cell B4 and highlight the block of cells from B4 to B12. Click the **Edit** command again but this time select **Paste** from the drop-down menu.

The cells B4 to B12 fill with the numbers ....

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The numbers 2·4 to 4·0 in intervals of 0·2

How has this happened? When you typed in the cell reference B2 into the formula in cell B3, the spreadsheet understood this to mean *the contents of the cell immediately above current cell B3*. When the formula is copied into cell B4 it means *the contents of the cell immediately above current cell B4*. Entered in this way the address B2 is a *relative address*. On the other hand, when you typed in $F$1 the spreadsheet understood this to mean the contents of cell F1 and that meaning remains when it is copied – the dollar signs indicate an *absolute address*. So as you move down the column the contents of a cell contain the contents of the cell immediately above it plus the contents of cell F1. You will shortly see the advantages of all this.

For now, place the cell highlight in cell C1 and enter the letter y – this column is going to contain the computed y-values against the corresponding x-values in column B. Place the cell highlight in cell C2 and enter the number 5 – the initial y-value. Before we can compute the y-values in column C we need to be able to tabulate the values of y' – the derivatives of y. Place the cell highlight in cell D1 and enter y' – this column will contain the values of the derivatives of y against the corresponding x-values. Cell D2 will contain the initial value of y' which can be computed from the equation

$$y' = 2(1 + x) - y$$

When $x = x_0 = 2$ and $y = y_0 = 5$ then

$$y'_0 = 2(1 + x_0) - y_0 = 2(1 + 2) - 5 = 1$$

so place the cell highlight in cell D2 and enter the formula

= 2 * (1 + B2) – C2 (B2 contains x_0 and C2 contains y_0)

The number 1 appears in cell D2. We need to copy this formula down the y' column. Place the cell highlight in cell D2, click **Edit** and select **Copy**. Now place the cell highlight in cell D3 and highlight the block of cells D3 to D12. Click the **Edit** command again and select **Paste**.

The cells D3 to D12 fill with ....

The numbers 6·4 to 10·0 in intervals of 0·4

Because the cells in the C2 column are currently empty, these values are just 2 ∗ (1 + B2) – 0.

Now, to compute the y-values we use the equation $y_1 = y_0 + h(y')_0$. Place the cell highlight in cell C3 and enter the formula

= C2 + \$F\$1 ∗ D2 (C2 contains y_0, F1 contains h and D2 contains $(y')_0$)

and the number 5·2 appears. That is, $y_1 = 5 + (0{\cdot}2)(1) = 5{\cdot}2$. This now completes the sequence of operations required to find y_1. To find the values of $y_2 = y(x_2) = y(2{\cdot}4)$ this sequence is repeated and, to ensure this, all that remains is to copy the formula in cell C3 into cells C4 to C12. So do this to reveal the following display

n	x	y	y′	0·2
0	2	5	1	
1	2·2	5·2	1·2	
2	2·4	5·44	1·36	
3	2·6	5·712	1·488	
4	2·8	6·0096	1·5904	
5	3	6·32768	1·67232	
6	3·2	6·662144	1·737856	
7	3·4	7·0097152	1·7902848	
8	3·6	7·36777216	1·83222784	
9	3·8	7·734217728	1·865782272	
10	4	8·107374182	1·892625818	

Now that was a lot easier than all that arithmetic manipulation by hand, wasn't it? We can tidy this display up by using the **Format** command and by using the various options on the tool bars to change the column widths and to display the numbers in a regular format of 10 decimal places to produce a display that is easier to read.

Next frame

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n	x	y	y′	*h*=0·2
0	2·0	5·0000000000	1·0000000000	
1	2·2	5·2000000000	1·2000000000	
2	2·4	5·4400000000	1·3600000000	
3	2·6	5·7120000000	1·4880000000	
4	2·8	6·0096000000	1·5904000000	
5	3·0	6·3276800000	1·6723200000	
6	3·2	6·6621440000	1·7378560000	
7	3·4	7·0097152000	1·7902848000	
8	3·6	7·3677721600	1·8322278400	
9	3·8	7·7342177280	1·8657822720	
10	4·0	8·1073741824	1·8926258176	

Notice that we have added ***h*=** in cell E1 and justified it to the right and then justified the number 0·2 in F2 to the left so that together they read as an equation. The advantage of isolating the step value 0·2 in cell F1, as we have done, is that we can change the value and immediately see the effects on the calculations. For example, if the contents of F1 are changed to 0·1 the display changes automatically to

n	x	y	y′	*h*=0·1
0	2·0	5·0000000000	1·0000000000	
1	2·1	5·1000000000	1·1000000000	
2	2·2	5·2100000000	1·1900000000	
3	2·3	5·3290000000	1·2710000000	
4	2·4	5·4561000000	1·3439000000	
5	2·5	5·5904900000	1·4095100000	
6	2·6	5·7314410000	1·4685590000	
7	2·7	5·8782969000	1·5217031000	
8	2·8	6·0304672100	1·5695327900	
9	2·9	6·1874204890	1·6125795110	
10	3·0	6·3486784401	1·6513215599	

Notice that the different values of h produce different corresponding values in the tables. For example, for $h = 0{\cdot}2$ we find that $y(3{\cdot}0) = 6{\cdot}327\,680\,0000$ whereas for $h = 0{\cdot}1$ we have $y(3{\cdot}0) = 6{\cdot}348\,678\,4401$. The smaller the value of h then, the smaller the errors in the calculation – we shall see this demonstrated explicitly in the next frame.

Go to the next frame

The exact value and the errors

The differential equation

$$y' = 2(1 + x) - y$$

can be solved using the integration factor method (see *Engineering Mathematics, Fifth Edition*, Programme 24) to give the solution

$$y = 2x + e^{2-x}$$

We can programme this into the spreadsheet to compare the exact solution with the solution obtained numerically and compute the actual errors. Place the cell highlight in cell E1 and highlight cells E1 and F1. Click **Insert** on the Command bar and select **Columns**. Immediately two new columns appear. Notice that the numbers in the display do not change despite the fact that the h-value of 0·2 has moved from F1 to H1 – all the formulas in the spreadsheet will have automatically adjusted themselves. You can check this by highlighting a cell with a formula in it to see the change.

In cell E1 enter the word **Exact** and in cell F1 enter **Errors (%)**. In cell E2 enter the right-hand side of the equation $y = 2x + e^{2-x}$ by using the formula

= 2 * B2 + EXP(2 – B2) (the EXP stands for the exponential function)

and copy this into the block of cells E3 to E12. In cell F2 enter the formula for the error

= (E2 – C2) * 100/E2 (the error as a percentage of the exact value)

and copy this into the block of cells F3 to F12 to produce the following display

n	x	y	y′	Exact	Errors (%)	h=0·2
0	2·0	5·0000000000	1·0000000000	5·0000000000	0·00	
1	2·2	5·2000000000	1·2000000000	5·2187307531	0·36	
2	2·4	5·4400000000	1·3600000000	5·4703200460	0·55	
3	2·6	5·7120000000	1·4880000000	5·7488116361	0·64	
4	2·8	6·0096000000	1·5904000000	6·0493289641	0·66	
5	3·0	6·3276800000	1·6723200000	6·3678794412	0·63	
6	3·2	6·6621440000	1·7378560000	6·7011942119	0·58	
7	3·4	7·0097152000	1·7902848000	7·0465969639	0·52	
8	3·6	7·3677721600	1·8322278400	7·4018965180	0·46	
9	3·8	7·7342177280	1·8657822720	7·7652988882	0·40	
10	4·0	8·1073741824	1·8926258176	8·1353352832	0·34	

▶

Change the value of h to 0·1 and produce the following display

n	x	y	y′	Exact	Errors (%)	h=0·1
0	2·0	5·0000000000	1·0000000000	5·0000000000	0·00	
1	2·1	5·1000000000	1·1000000000	5·1048374180	0·09	
2	2·2	5·2100000000	1·1900000000	5·2187307531	0·17	
3	2·3	5·3290000000	1·2710000000	5·3408182207	0·22	
4	2·4	5·4561000000	1·3439000000	5·4703200460	0·26	
5	2·5	5·5904900000	1·4095100000	5·6065306597	0·29	
6	2·6	5·7314410000	1·4685590000	5·7488116361	0·30	
7	2·7	5·8782969000	1·5217031000	5·8965853038	0·31	
8	2·8	6·0304672100	1·5695327900	6·0493289641	0·31	
9	2·9	6·1874204890	1·6125795110	6·2065696597	0·31	
10	3·0	6·3486784401	1·6513215599	6·3678794412	0·30	

When $h = 0{\cdot}2$ the error in $y(3{\cdot}0)$ is 0·63% whereas when $h = 0{\cdot}1$ the error in $y(3{\cdot}0)$ is 0·30%.

The smaller the value of h the ....

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smaller the error

Having completed your first spreadsheet you can now use it as a template for similar problems.

To avoid losing the work that you have already done, save your spreadsheet under some suitable name. When that is complete, highlight all the cells from A1 to G12 and copy them onto the clipboard using the **Edit-Copy** sequence of commands. Now click the **Sheet 2** tab at the bottom of your spreadsheet to reveal a blank worksheet. Place the cell highlight in cell A1, click **Edit** and select **Paste**. The entire contents of **Sheet 1** are now copied to **Sheet 2** in readiness for editing to accommodate a new problem.

So let's look at another example.

Example 2

Obtain a numerical solution of the equation

$$\frac{dy}{dx} = 1 + x - y$$

with the initial condition that $y = 2$ at $x = 1$, for the range $x = 1{\cdot}0(0{\cdot}2)3{\cdot}0$, that is from $x = 1{\cdot}0$ to $x = 3{\cdot}0$ with step length $x = 0{\cdot}2$.

As initial conditions, we have

$x_0 = \ldots\ldots\ldots\ldots$ and $y_0 = \ldots\ldots\ldots\ldots$

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$$\boxed{x_0 = 1, \quad y_0 = 2}$$

Because

$x_0 = 1$ and $y_0 = 2$ are given initial conditions.

These values can now be inserted into the spreadsheet in cells ....

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$$\boxed{x_1 = 1 \text{ in B2}, \quad y_0 = 2 \text{ in C2}}$$

Notice how the numbers in column B have changed to accommodate the new sequence of x-values. The contents of the cells in column C do not need to be changed as they refer to the equation

$$y_1 = y_0 + h(y')_0$$

which is the same in this spreadsheet as it was in the previous spreadsheet. The contents of column D do have to be changed because they currently refer to the equation to be solved in the previous problem. The equation to be solved here is

$$y' = 1 + x - y$$

so in cell D3 the contents need to be changed to ....

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$$\boxed{= 1 + \text{B2} - \text{C2}}$$

This formula must then be copied into cells C3 to C12. Finally, the **Exact** column needs to be amended to reflect the exact solution to this equation, which is again found by using the integration factor method as

$$y = x + e^{1-x}$$

So, in E2, enter the formula ....

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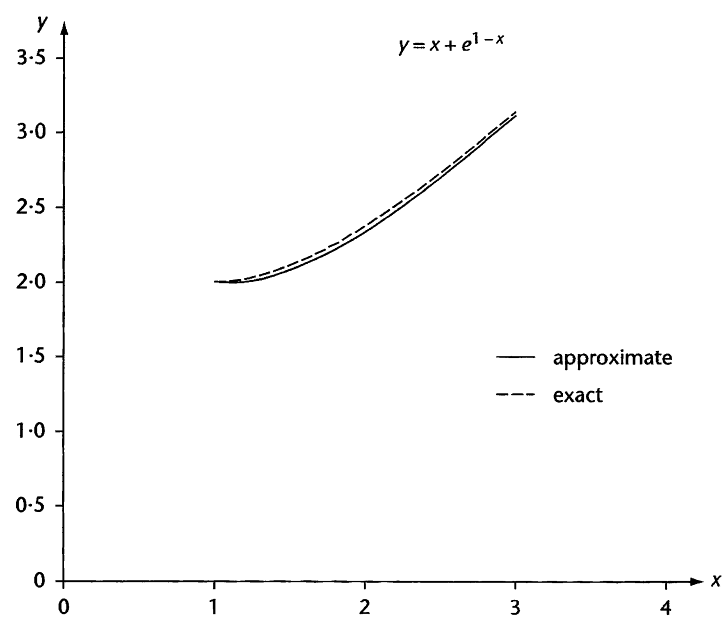
$$\boxed{= \text{B2} + \text{EXP}(1 - \text{B2})}$$

This formula needs to be copied into cells E3 to E12. This completes the editing of the spreadsheet to reflect the new problem to give the display

n	**x**	**y**	**y′**	**Exact**	**Errors (%)**	h=0·2
0	1·0	2·0000000000	0·0000000000	2·0000000000	0·00	
1	1·2	2·0000000000	0·2000000000	2·0187307531	0·93	
2	1·4	2·0400000000	0·3600000000	2·0703200460	1·46	
3	1·6	2·1120000000	0·4880000000	2·1488116361	1·71	
4	1·8	2·2096000000	0·5904000000	2·2493289641	1·77	
5	2·0	2·3276800000	0·6723200000	2·3678794412	1·70	
6	2·2	2·4621440000	0·7378560000	2·5011942119	1·56	
7	2·4	2·6097152000	0·7902848000	2·6465969639	1·39	
8	2·6	2·7677721600	0·8322278400	2·8018965180	1·22	
9	2·8	2·9342177280	0·8657822720	2·9652988882	1·05	
10	3·0	3·1073741824	0·8926258176	3·1353352832	0·89	

A plot of the graph of y against x for both the computed value and the exact value looks as follows



Graphical interpretation of Euler's method

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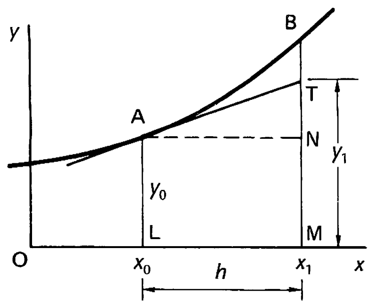

If AT is the tangent to the curve at A,

then $\dfrac{NT}{AN} = \left[\dfrac{dy}{dx}\right]_{x=x_0} = (y')_0$

$\dfrac{NT}{h} = (y')_0 \qquad \therefore \ NT = h(y')_0$

$\therefore$ At $x = x_1$, $MT = y_0 + h(y')_0$

By Euler's relationship, $y_1 = y_0 + h(y')_0$ i.e. MT.

The difference between the calculated value of y, i.e. MT, and the actual value of the function y, i.e. MB, at $x = x_1$, is indicated by TB. This error can be considerable, depending on the curvature of the graph and the size of the interval h. It is inherent to the method and corresponds to the truncation of the Taylor's series after the second term.

Euler's method, then

(a) is simple in procedure

(b) is lacking in accuracy, especially away from the starter values of the initial conditions

(c) is of use only for very small values of the interval h.

In spite of its practical limitations, it is the foundation of several more sophisticated methods and hence it is worthy of note.

Here is one more example to work on your own.

Example 3

Obtain the solution of $\dfrac{dy}{dx} = x + y$ with the initial condition that $y = 1$ at $x = 0$, for the range $x = 0(0{\cdot}1)0{\cdot}5$.

By using a previously constructed spreadsheet as a template, the solution is

....

The function values are given in the next frame

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n	x	y	y′	Exact	Errors (%)	$h = 0{\cdot}1$
0	0·0	1·0000000000	1·0000000000	1·0000000000	0·00	
1	0·1	1·1000000000	1·2000000000	1·1103418362	0·93	
2	0·2	1·2200000000	1·4200000000	1·2428055163	1·84	
3	0·3	1·3620000000	1·6620000000	1·3997176152	2·69	
4	0·4	1·5282000000	1·9282000000	1·5836493953	3·50	
5	0·5	1·7210200000	2·2210200000	1·7974425414	4·25	
6	0·6	1·9431220000	2·5431220000	2·0442376008	4·95	
7	0·7	2·1974342000	2·8974342000	2·3275054149	5·59	
8	0·8	2·4871776200	3·2871776200	2·6510818570	6·18	
9	0·9	2·8158953820	3·7158953820	3·0192062223	6·73	
10	1·0	3·1874849202	4·1874849202	3·4365636569	7·25	

Because

The initial conditions are entered as

0 in cell B2 (the initial x-value)
1 in cell C2 (the initial y-value)
0·1 in cell H1 (the x step length)

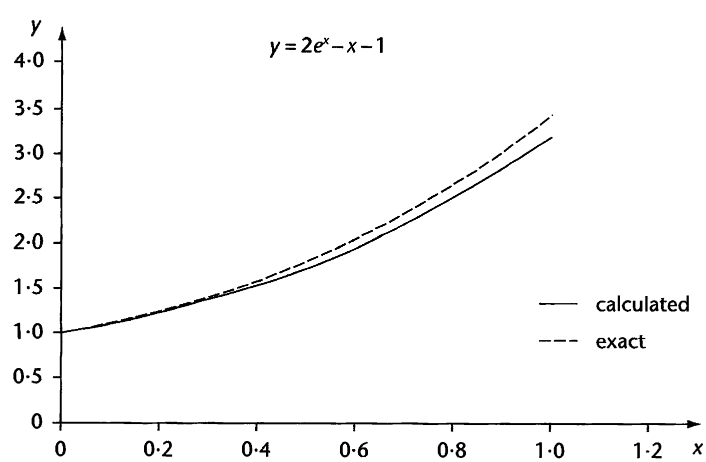
The formulas are entered as

=B2+C2 in cell D2, copied into cells D3 to D12
(the successive y'-values)
=C2+$H$1*D2 in cell C3 copied into cells C4 to C12
(the successive y-values)

The exact solution found by using the integration factor method is $y = 2e^x - x - 1$ and so

=2*EXP(B2) – B2 – 1 is entered into cell E2 and copied into cells E3 to E12

Notice how the errrors here are significant, which is very evident from the graphs of the computed values and the exact values.



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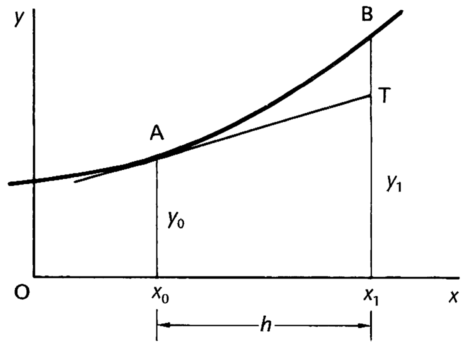
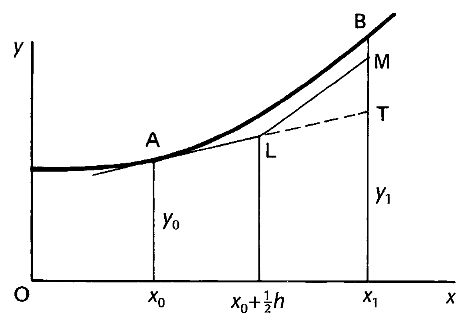
The Euler–Cauchy method – or the improved Euler method



In Euler's method, we use the slope $(y')_0$ at A (x_0, y_0) across the whole interval h to obtain an approximate value of y_1 at B. TB is the resulting error in the result.



In the Euler–Cauchy method, we use the slope at A (x_0, y_0) across half the interval and then continue with a line whose slope approximates to the slope of the curve at x_1.

Let $\bar{\bar{y}}_1$ be the y-value of the point at T.

The error (MB) in the result is now considerably less than the error (TB) associated with the basic Euler method and the calculated results will accordingly be of greater accuracy.

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Euler–Cauchy calculations

The steps in the Euler–Cauchy method are as follows.

1 We start with the given equation $y' = f(x, y)$ with the initial condition that at $x = x_0$, $y = y_0$. We have to determine function values for $x = x_0(h)x_n$.

2 From the equation and the initial condition we obtain $(y')_0 = f(x_0, y_0)$.

3 Knowing x_0, y_0, $(y')_0$ and h, we then evaluate

(a) $x_1 = x_0 + h$

(b) the auxiliary value of y, denoted by $\bar{\bar{y}}$ where $\bar{\bar{y}}_1 = y_0 + h(y')_0$.This is the same step as in Euler's method.

(c) Then $y_1 = y_0 + \frac{1}{2}h\{(y')_0 + f(x_1, \bar{\bar{y}}_1)\}$

Note that $f(x_1, y_1)$ is the right-hand side of the given equation with x and y replaced by the calculated values of x_1 and $\bar{\bar{y}}_1$.

(d) Finally $(y')_1 = f(x_1, y_1)$.

We have thus evaluated x_1, y_1 and $(y')_1$.

The whole process is then repeated, the calculated values of x_1, y_1 and $(y')_1$ becoming the starter values x_0, y_0, $(y')_0$ for the next stage.

Make a note of the relationships above. We shall be using them quite often.

Then on to the next frame for an example of their use

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Example 1

Apply the Euler–Cauchy method to solve the equation

$$y' = x + y$$

with the initial condition that at $x = 0$, $y = 1$, for the range $x = 0(0{\cdot}1)1{\cdot}0$.

We proceed as before by copying our template solution to a new worksheet. Before we continue we need to decide what the entries are going to be in our spreadsheet.

1 We are going to have to enter new initial conditions, so

Enter 0 in cell B2	that is $x_0 = 0$
Enter 1 in cell C2	that is $y_0 = 1$
Enter 0·1 in cell H1	this is the x step length

2 The equation to be solved is $y' = x + y$, so enter the formula

=B2+C2 in cell D2 and copy the contents of D2 into cells D3 to D12

3 The Euler–Cauchy method tells us that

$$y_1 = y_0 + \frac{1}{2}h\{(y')_0 + f(x, \bar{\bar{y}}_1)\}$$

where $\bar{\bar{y}}_1 = y_0 + h(y')_0$ so that

$$f(x_1, \bar{\bar{y}}_1) = x_1 + \bar{\bar{y}}_1 = x_1 + y_0 + h(y')_0$$

Therefore $y_1 = \dots\dots\dots\dots$

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$$\boxed{y_1 = y_0 + \frac{1}{2}h\{x_1 + y_0 + (1+h)(y')_0\}}$$

Because

By replacing $f(x_1, \bar{\bar{y}}_1)$ with $x_1 + y_0 + h(y')_0$ in the expression

$y_1 = y_0 + \frac{1}{2}h\{(y')_0 + f(x_1, \bar{\bar{y}}_1)\}$

we find that

$$\begin{aligned} y_1 &= y_0 + \tfrac{1}{2}h\{(y')_0 + x_1 + y_0 + h(y')_0\} \\ &= y_0 + \tfrac{1}{2}h\{x_1 + y_0 + (1+h)(y')_0\} \end{aligned}$$

In cell C3 enter the formula ....

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=C2+(0·5)*$H$1*(B3+C2+(1+$H$1)*D2)

Because

y_0 is in cell C2, h is in cell H1, x_1 is in cell B3 and $(y')_0$ is in cell D2.

Copy the contents of cell C3 into cells C4 to C12.

4 Finally, for comparison purposes, the exact solution of this equation is $y = 2e^x - x - 1$ and this is

entered into E2 by the formula ....
and copied into cells ....

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=2*EXP(B2) − B2 − 1 and copied into cells E3 to E12

The resulting display looks as follows

n	**x**	**y**	**y′**	**Exact**	**Errors (%)**	***h*** = 0·1
0	0·0	1·0000000000	1·0000000000	1·0000000000	0·00	
1	0·1	1·1100000000	1·2100000000	1·1103418362	0·03	
2	0·2	1·2420500000	1·4420500000	1·2428055163	0·06	
3	0·3	1·3984652500	1·6984652500	1·3997176152	0·09	
4	0·4	1·5818041013	1·9818041013	1·5836493953	0·12	
5	0·5	1·7948935319	2·2948935319	1·7974425414	0·14	
6	0·6	2·0408573527	2·6408573527	2·0442376008	0·17	
7	0·7	2·3231473748	3·0231473748	2·3275054149	0·19	
8	0·8	2·6455778491	3·4455778491	2·6510818570	0·21	
9	0·9	3·0123635233	3·9123635233	3·0192062223	0·23	
10	1·0	3·4281616932	4·4281616932	3·4365636569	0·24	

▶

Comparing these results with the same equation being solved by the Euler method demonstrates how much more accurate the Euler–Cauchy method is, as can be seen from the following table of comparative errors

x	Euler	Euler–Cauchy
0·0	0·00	0·00
0·1	0·93	0·03
0·2	1·84	0·06
0·3	2·69	0·09
0·4	3·50	0·12
0·5	4·25	0·14
0·6	4·95	0·17
0·7	5·59	0·19
0·8	6·18	0·21
0·9	6·73	0·23
1·0	7·25	0·24

Next frame

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Now for another example, but before that, complete the following without reference to your notes – if possible. In the Euler–Cauchy method the relevant relationships are

$$x_1 = \dots\dots\dots\dots$$
$$\bar{\bar{y}}_1 = \dots\dots\dots\dots$$
$$y_1 = \dots\dots\dots\dots$$
$$(y')_1 = \dots\dots\dots\dots$$

Next frame

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$$\boxed{\begin{aligned} x_1 &= x_0 + h \\ \bar{\bar{y}}_1 &= y_0 + h(y')_0 \\ y_1 &= y_0 + \frac{1}{2}h\{(y')_0 + f(x_1, \bar{\bar{y}}_1)\} \\ (y')_1 &= f(x_1, y_1) \end{aligned}}$$

Example 2

Determine a numerical solution of the equation $y' = 2(1 + x) - y$ with the initial condition that $y = 5$ when $x = 2$, for the range 2·0(0·2)4·0. Try this one yourself.

The exact solution is given as $y = 2x + e^{-2x}$
and the final display of results is ....

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n	x	y	y′	Exact	Errors (%)	$h = 0{\cdot}2$
0	2·0	5·0000000000	1·0000000000	5·0000000000	0·00	
1	2·2	5·2200000000	1·1800000000	5·2187307531	−0·02	
2	2·4	5·4724000000	1·3276000000	5·4703200460	−0·04	
3	2·6	5·7513680000	1·4486320000	5·7488116361	−0·04	
4	2·8	6·0521217600	1·5478782400	6·0493289641	−0·05	
5	3·0	6·3707398432	1·6292601568	6·3678794412	−0·04	
6	3·2	6·7040066714	1·6959933286	6·7011942119	−0·04	
7	3·4	7·0492854706	1·7507145294	7·0465969639	−0·04	
8	3·6	7·4044140859	1·7955859141	7·4018965180	−0·03	
9	3·8	7·7676195504	1·8323804496	7·7652988882	−0·03	
10	4·0	8·1374480313	1·8625519687	8·1353352832	−0·03	

Because

1 The initial conditions are entered as

Enter 2 in cell B2 (that is $x_0 = 2$); enter 5 in cell C2 (that is $y_0 = 5$)
Enter 0·2 in cell H1 (this is the x step length)

2 The equation to be solved is $y' = 2(1 + x) - y$, so enter the formula

=2∗(1+B2)−C2 in cell D2 and copy the contents of D2 into cells D3 to D12

3 The Euler–Cauchy method tells us that

$y_1 = y_0 + \frac{1}{2}h\{(y')_0 + f(x_1, \bar{\bar{y}}_1)\}$ where $\bar{\bar{y}}_1 = y_0 + h(y')_0$ so that

$f(x_1, \bar{\bar{y}}_1) = 2(1 + x_1) - \bar{\bar{y}}_1 = 2(1 + x_1) - y_0 - h(y')_0$ therefore

$y_1 = y_0 + \frac{1}{2}h\{(y')_0 + 2(1 + x_1) - y_0 - h(y')_0\}$ that is

$y_1 = y_0 + \frac{1}{2}h\{2(1 + x_1) - y_0 + (1 - h)(y')_0\}$

This is accommodated by the formula in C3 (copied into cells C4 to C12)

=C2+(0·5)∗$H$1∗(2∗(1+B3) − C2+(1 − $H$1)∗D2)

4 Finally the exact solution $y = 2x + e^{-2x}$ is entered into cell E2 as =2∗B2+EXP(−2∗B2) and copied into cells E3 to E12.

Refer to Frame 19 for a comparison of errors between this method and the Euler method. Then another example for you to try just to make sure you are clear about the processes involved.

Next frame

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Example 3

Solve the equation $y' = y^2 + xy$ with initial condition that at $x = 1$, $y = 1$, for the range $x = 1{\cdot}0(0{\cdot}1)1{\cdot}7$. Use the Euler–Cauchy method and work to 6 places of decimals.

The solution is ....

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n	x	y	y′	$h = 0{\cdot}1$
0	1·0	1·000000	2·000000	
1	1·1	1·238000	2·894444	
2	1·2	1·591023	4·440583	
3	1·3	2·152410	7·431004	
4	1·4	3·145846	14·300528	
5	1·5	5·251007	35·449581	
6	1·6	11·595613	153·011211	
7	1·7	57·704110	3427·861242	

Because

1 The initial conditions are entered as

Enter 1 in cell B2 (that is $x_0 = 1$); enter 1 in cell C2 (that is $y_0 = 1$)
Enter 0·1 in cell H1 (this is the x step length)

2 The equation to be solved is $y' = y^2 + xy$, so

Enter the formula =C2^2+B2*C2 in cell D2 and copy the contents of D2 into cells D3 to D9. Note that C2^2=C2*C2 – the 'hat' indicates raising to a power.

3 The Euler-Cauchey method tell us that

$$y_1 = y_0 + \frac{1}{2}h\{(y')_0 + f(x_1, \bar{\bar{y}}_1)\} \quad \text{where } \bar{\bar{y}}_1 = y_0 + h(y')_0 \text{ so that}$$

$$f(x_1, \bar{\bar{y}}_1) = \bar{\bar{y}}_1^{\,2} + x_1\bar{\bar{y}}_1 = (y_0 + h(y')_0)^2 + x_1(y_0 + h(y')_0) \text{ therefore}$$

$$y_1 = y_0 + \frac{1}{2}h\left\{(y')_0 + (y_0 + h(y')_0)^2 + x_1(y_0 + h(y')_0)\right\}$$

This is accommodated by the formula in C3 (copied into cells C4 to C9)

=C2+(0·5)*$F$1*(D2+(C2+$F$1*D2))^2+
B3*(C2+$F$1*D2))

The table shows that as x increases, the computed values of y and its derivative increase dramatically. This is an indication that the exact solution increases without bound near to the larger values of x considered, so bringing the accuracy of these computed values into question. This emphasises the importance of checking every method against a known solution so as to form some idea of the method's accuracy. However, all numerical methods produce significant accuracies whenever the exact solution diverges in this way.

Runge–Kutta method

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The Runge–Kutta method for solving first-order differential equations is widely used and affords a high degree of accuracy. It is a further step-by-step process where a table of function values for a range of values of x is accumulated. Several intermediate calculations are required at each stage, but these are straightforward and present little difficulty.

In general terms, the method is as follows.

To solve $y' = f(x, y)$ with initial condition $y = y_0$ at $x = x_0$, for a range of values of $x = x_0(h)x_n$.

Starting as usual with $x = x_0$, $y = y_0$, $y' = (y')_0$ and h, we have

$$x_1 = x_0 + h$$

Finding y_1 requires four intermediate calculations

$$k_1 = hf(x_0, y_0) = h(y')_0$$
$$k_2 = hf(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}k_1)$$
$$k_3 = hf(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}k_2)$$
$$k_4 = hf(x_0 + h,\ y_0 + k_3)$$

The increment Δy_0 in the y-values from $x = x_0$ to $x = x_1$ is then

$$\Delta y_0 = \tfrac{1}{6}\{k_1 + 2k_2 + 2k_3 + k_4\}$$

and finally $\quad y_1 = y_0 + \Delta y_0.$

We shall be using these repeatedly, so make a note of them for future reference. Then let us see an example

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Example 1

Find the numerical solution of $y' = x + y$ using the Runge–Kutta method with $y = 1$ and $x = 0$ for values in the range $x = 0(0{\cdot}1)1{\cdot}0$.

We shall proceed with the solution of this differential equation using a spreadsheet in much the same manner as before. However, we are going to require a different structure in order to accommodate the four variables k_i for $i = 1, 2, 3, 4$. The structure we shall use is headed by

	A	B	C	D	E	F	G	H	I
1	**n**	**x**	**k1**	**k2**	**k3**	**k4**	**y**	**y′**	***h*=**

where the value of h is held in cell J1.

1 Enter the values 0 to 10 in column A from A2 to A12 using the **Edit-Fill-Series** sequence of commands. These are the iteration numbers.
2 Enter the x step value of 0·1 in cell J1.
3 Enter the initial value of x in cell B2 as 0 and in B3 enter the formula =B2+$J$1. Now copy the contents of B3 into cells B4 to B12.
4 Enter the initial value of y in cell G2 as 1.

We can now progressively enter the table of values from the left.

5 $k_1 = hf(x_0, y_0) = h(y')_0$ – the y'-values are in column H, so in cell C2 enter the formula =$J$1 * H2. Copy the contents of C2 into cells C3 to C12.
6 $k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = h\left(x_0 + \frac{1}{2}h + y_0 + \frac{1}{2}k_1\right)$, so in cell D2 enter the formula =$J$1 * (B2+0·5 * $J$1+G2+0·5 * C2). Copy the contents of D2 into cells D3 to D12.
7 $k_3 = hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2) = h(x_0 + \frac{1}{2}h + y_0 + \frac{1}{2}k_2)$, so in cell E2 enter the formula =$J$1 * (B2+0·5 * $J$1+G2+0·5 * D2). Copy the contents of E2 into cells E3 to E12.
8 $k_4 = hf(x_0 + h, y_0 + k_3) = h(x_0 + h + y_0 + k_3)$, so in cell F2 enter the formula = $J$1 * (B2 + $J$1 + G2 + E2). Copy the contents of F2 into cells F3 to F12.
9 $y_1 = y_0 + \frac{1}{6}\{k_1 + 2k_2 + 2k_3 + k_4\}$, so in cell G3 enter the formula =G2+(1/6) * (C2+2 * D2+2 * E2+F2). Copy the contents of G3 into cells G4 to G12.
10 $y' = x + y$, so in H2 enter the formula =B2+G2. Copy the contents of H2 into cells H3 to H12.

The results are displayed in the next frame

n	x	k1	k2	k3	k4	y	y′	$h = 0{\cdot}1$
0	0·0	0·1000000	0·1100000	0·1105000	0·1210500	1·0000000	1·0000000	
1	0·1	0·1210342	0·1320859	0·1326385	0·1442980	1·1103417	1·2103417	
2	0·2	0·1442805	0·1564945	0·1571052	0·1699910	1·2428051	1·4428051	
3	0·3	0·1699717	0·1834703	0·1841452	0·1983862	1·3997170	1·6997170	
4	0·4	0·1983648	0·2132831	0·2140290	0·2297677	1·5836485	1·9836485	
5	0·5	0·2297441	0·2462313	0·2470557	0·2644497	1·7974413	2·2974413	
6	0·6	0·2644236	0·2826448	0·2835558	0·3027792	2·0442359	2·6442359	
7	0·7	0·3027503	0·3228878	0·3238947	0·3451398	2·3275033	3·0275033	
8	0·8	0·3451079	0·3673633	0·3684761	0·3919555	2·6510791	3·4510791	
9	0·9	0·3919203	0·4165163	0·4177461	0·4436949	3·0192028	3·9192028	
10	1·0	0·4436559	0·4708387	0·4721979	0·5008757	3·4365595	4·4365595	

with the following errors

n	x	Exact	Error (%)	Error (%)
0	0·0	1·0000000	0·0000000	0·00
1	0·1	1·1103418	0·0000153	0·93
2	0·2	1·2428055	0·0000301	1·84
3	0·3	1·3997176	0·0000444	2·69
4	0·4	1·5836494	0·0000578	3·50
5	0·5	1·7974425	0·0000703	4·25
6	0·6	2·0442376	0·0000820	4·95
7	0·7	2·3275054	0·0000929	5·59
8	0·8	2·6510819	0·0001030	6·18
9	0·9	3·0192062	0·0001124	6·73
10	1·0	3·4365637	0·0001213	7·25

The column to the far right contains the errors using the Euler method and, as you can see, the Runge–Kutta method provides a significant improvement in accuracy.

Now, without reference to your notes, complete the following expressions for

$$k_1 = \dots\dots\dots\dots$$
$$k_2 = \dots\dots\dots\dots$$
$$k_3 = \dots\dots\dots\dots$$
$$k_4 = \dots\dots\dots\dots$$
$$\Delta y_0 = \dots\dots\dots\dots$$
$$y_1 = \dots\dots\dots\dots$$

It speeds up your working if you can remember them.

41

$$
\begin{aligned}
k_1 &= h(y')_0 \\
k_2 &= hf\left(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}k_1\right) \\
k_3 &= hf\left(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}k_2\right) \\
k_4 &= hf(x_0 + h,\ y_0 + k_3) \\
\Delta y_0 &= \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\
y_1 &= y_0 + \Delta y_0
\end{aligned}
$$

With those in mind, let us move on to a further example. Next frame

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Example 2

Solve $y' = \sqrt{x^2 + y}$ for $x = 0(0{\cdot}2)2{\cdot}0$ given that at $x = 0$, $y = 0{\cdot}8$.

Using the spreadsheet for the previous example as a template for this example. The solution is ....

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n	x	k1	k2	k3	k4	y	y′	*h*=0·2
0	0·0	0·1788854	0·1896779	0·1902460	0·2030021	0·8000000	0·8944272	
1	0·2	0·2030063	0·2174206	0·2180825	0·2339548	0·9902892	1·0150316	
2	0·4	0·2339473	0·2510185	0·2516977	0·2698134	1·2082838	1·1697366	
3	0·6	0·2698011	0·2887709	0·2894271	0·3091435	1·4598160	1·3490055	
4	0·8	0·3091304	0·3294604	0·3300769	0·3509482	1·7490394	1·5456518	
5	1·0	0·3509358	0·3722562	0·3728285	0·3945492	2·0788983	1·7546790	
6	1·2	0·3945381	0·4165946	0·4171237	0·4394829	2·4515074	1·9726904	
7	1·4	0·4394732	0·4620889	0·4625781	0·4854274	2·8684170	2·1973659	
8	1·6	0·4854190	0·5084682	0·5089213	0·5321545	3·3307894	2·4270948	
9	1·8	0·5321472	0·5555390	0·5559599	0·5794989	3·8395148	2·6607358	
10	2·0	0·5794925	0·6031595	0·6035518	0·6273385	4·3952888	2·8974625	

Because

1. The initial conditions are entered as $x_0 = 0$ and $y_0 = 0{\cdot}8$. The x step length is entered as 0·2
2. The formula for the variable k_1 remains the same as = $J$1 * H2
3. The formula for the variable k_2 is changed to = $J$1 * (((B2 + 0·5 * $J$1)^2 + G2 + 0·5 * C2)^0·5)
4. The formula for the variable k_3 is changed to = $J$1 * (((B2 + 0·5 * $J$1)^2 + G2 + 0·5 * D2)^0·5)
5. The formula for the variable k_4 is changed to = $J$1 * (((B2 + $J$1)^2 + G2 + E2)^0·5)
6. The formula for y remains the same as = G2+(1/6) * (C2 + 2 * D2 + 2 * E2 + F2)
7. The formula for y' is changed to = (B2^2+G2)^0·5

That is it. Now move on to the next frame where we make a new start and apply similar methods to the solution of second-order differential equations by numerical methods.

Second-order differential equations

Euler second-order method

44

The first method we will deal with is really an extension of the Euler method for the first-order equations and is a direct application of a truncated form of Taylor's series. We anticipate, therefore, that the method will be relatively easy, but the results will not be accurate to a high degree.

Taylor's series:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots$$

Differentiating term by term with respect to x, we obtain

$$f'(x+h) = f'(x) + hf''(x) + \frac{h^2}{2!}f'''(x) + \frac{h^3}{3!}f''''(x) + \dots$$

If we neglect terms in $f'''(x)$ and subsequent terms in each of these two series, we have the approximations

$$f(x+h) \approx \dots\dots\dots\dots$$
$$f'(x+h) \approx \dots\dots\dots\dots$$

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$$\boxed{\begin{aligned} f(x+h) &\approx f(x) + hf'(x) + \frac{h^2}{2!}f''(x) \\ f'(x+h) &\approx f'(x) + hf''(x) \end{aligned}}$$

Although these are approximations, in practice we tend to write them with the 'equals' sign. Therefore, at $x = a$, these become

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

and $$\dots\dots\dots\dots\dots\dots\dots\dots$$

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$$\boxed{\begin{aligned} f(a+h) &= f(a) + hf'(a) + \frac{h^2}{2!}f''(a) \\ f'(a+h) &= f'(a) + hf''(a) \end{aligned}}$$

and these, with the notation we have previously used, can be written

$$y_1 = y_0 + h(y')_0 + \frac{h^2}{2!}(y'')_0$$
$$(y')_1 = (y')_0 + h(y'')_0$$

Thus, if x_0, y_0, $(y')_0$ and $(y'')_0$ are known, we can find an approximate value of y_1 at $x_1 = x_0 + h$.

Make a note of these two relationships: then we can apply them.

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Example

Solve the equation $y'' = xy' + y$ for $x = 0(0{\cdot}2)2{\cdot}0$ given that at $x = 0$, $y = 1$ and $y' = 0$.

We shall set about finding the numerical solution to this equation as we have done previously by using a spreadsheet. The headings for the sheet will be

	A	B	C	D	E	F	G	H
1	**n**	**x**	**y**	**y′**	**y″**	**Exact**	**Errors (%)**	***h*=**

The entries will then be

1 Column A contains the iteration number from 0 in A2 to 10 in A12.

2 Cell I1 contains the x step length which is 0·2.

3 Column B contains the successive x-values from 0·0 to 2·0 in steps of 0·2. The initial value of $x_0 = 0$ is entered into cell B2 and the formula =B2+$I$1 is entered into cell B3 and copied into cells B4 to B12.

4 Column C contains the computed y-values. The initial value of $y_0 = 1$ is entered into cell C2 and the equation

$$y_1 = y_0 + h(y')_0 + \frac{h^2}{2!}(y'')_0$$

is represented in cell C3 by the formula

=C2+$I$1*D2+($I$1^2)*E2/2

copied into cells C4 to C12.

5 Column D contains the computed y'-values. The initial value of $(y')_0 = 0$ is entered into cell D2 and the equation

$$(y')_1 = (y')_0 + h(y'')_0$$

is represented in cell D3 by the formula =D2+$I$1*E2 copied into cells D4 to D12.

6 Column E contains the y''-values which are obtained from the equation $y'' = xy' + y$ which is represented in cell E2 by the formula =B2*D2+C2 copied into cells E3 to E12.

7 Column F contains the values obtained from the exact solution which can be shown to be $y = e^{x^2/2}$. This is represented in cell F2 by the formula =EXP((B2^2)/2) copied into cells F3 to F12.

8 Column G contains the percentage errors. In cell G2 enter the formula =(F2 – C2)*100/F2 copied into cells G3 to G12.

Your spreadsheet should now look like the one on the next page (with the appropriate formatting to make it easier to read).

▶

n	**x**	**y**	**y′**	**y″**	**Exact**	**Errors (%)** $h = 0{\cdot}2$
0	0·0	1·0000000	0·0000000	1·0000000	1·0000000	0·00
1	0·2	1·0200000	0·2000000	1·0600000	1·0202013	0·02
2	0·4	1·0812000	0·4120000	1·2460000	1·0832871	0·19
3	0·6	1·1885200	0·6612000	1·5852400	1·1972174	0·73
4	0·8	1·3524648	0·9782480	2·1350632	1·3771278	1·79
5	1·0	1·5908157	1·4052606	2·9960763	1·6487213	3·51
6	1·2	1·9317893	2·0044759	4·3371604	2·0544332	5·97
7	1·4	2·4194277	2·8719080	6·4400989	2·6644562	9·20
8	1·6	3·1226113	4·1599278	9·7784957	3·5966397	13·18
9	1·8	4·1501667	6·1156269	15·1582952	5·0530903	17·87
10	2·0	5·6764580	9·1472859	23·9710299	7·3890561	23·18

You will notice that the errors are significant and grow dramatically as the value of x increases. The main cause of errors is ....

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the truncation of the Taylor's series on which the method is based

A greater degree of accuracy can be obtained by using the Runge–Kutta method for second-order differential equations, which is an extension of the method we have already used for first-order equations. As before, more intermediate calculations are required, but the reliability of results reflects the extra work involved.

Runge–Kutta method for second-order differential equations

Starting with the given equation $y'' = f(x, y, y')$ and initial conditions that at $x = x_0$, $y = y_0$ and $y' = (y')_0$, we can obtain the value of y_1 at $x_1 = x_0 + h$ as follows.

(a) We evaluate

$$k_1 = \tfrac{1}{2}h^2 f\{x_0,\ y_0,\ (y')_0\} = \tfrac{1}{2}h^2 (y'')_0$$

$$k_2 = \tfrac{1}{2}h^2 f\left\{x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}h(y')_0 + \tfrac{1}{4}k_1,\ (y')_0 + \frac{k_1}{h}\right\}$$

$$k_3 = \tfrac{1}{2}h^2 f\left\{x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}h(y')_0 + \tfrac{1}{4}k_1,\ (y')_0 + \frac{k_2}{h}\right\}$$

$$k_4 = \tfrac{1}{2}h^2 f\left\{x_0 + h,\ y_0 + h(y')_0 + k_3,\ (y')_0 + \frac{2k_3}{h}\right\}$$

(b) From these results, we then determine

$$P = \tfrac{1}{3}\{k_1 + k_2 + k_3\}$$

$$Q = \tfrac{1}{3}\{k_1 + 2k_2 + 2k_3 + k_4\}$$

▶

(c) Finally, we have

$$x_1 = x_0 + h$$
$$y_1 = y_0 + h(y')_0 + P$$
$$(y')_1 = (y')_0 + \frac{Q}{h}$$

It is not as complicated as it looks at first sight. Copy down this list of relationships for reference when dealing with some examples that follow.

Then move on

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Note the following

1 Four evaluations for k are required to determine a single new point on the solution curve.
2 The method is self-starting in that no preliminary calculations are required. The equation and initial conditions are sufficient to provide the next point on the curve.
3 As with the Runge–Kutta method for first-order equations, the method contains no self-correcting element or indication of any error involved.

Example 1

Use the Runge–Kutta method to solve the equation $y'' = xy' + y$ for $x = 0{\cdot}0(0{\cdot}2)2{\cdot}0$ given that at $x = 0$, $y = 1$ and $y' = 0$.

This is the same problem that we have just encountered and in due course we shall compare results. As expected, we shall use a spreadsheet to derive the solution. The headings for the sheet this time will be

	A	B	C	D	E	F	G	H	I	J	K	L
1	**n**	**x**	**k1**	**k2**	**k3**	**k4**	**P**	**Q**	**y**	**y′**	**y″**	***h*=**

The entries will then be

1 Column A contains the iteration number from 0 in A2 to 10 in A12.
2 Cell M1 contains the x step length which is 0·2.
3 Column B contains the successive x-values from 0·0 to 2·0 in steps of 0·2. The initial value of $x_0 = 0$ is entered into cell B2 and the formula =B2+$M$1 is entered into cell B3 and copied into cells B4 to B12.
4 Column C contains the computed k_1-values and the equation $k_1 = \frac{1}{2}h^2(y'')_0$ is represented in cell C2 by the formula ....

50

=(0·5) * ($M$1^2) * K2

The contents of cell C2 are then copied into cells C3 to C12.

5 Column D contains the computed k_2-values and the equation

$$k_2 = \tfrac{1}{2}h^2 f(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}h(y')_0 + \tfrac{1}{4}k_1,\ (y')_0 + k_1/h)$$
$$= \tfrac{1}{2}h^2\big((x_0 + \tfrac{1}{2}h)\big((y')_0 + k_1/h\big) + y_0 + \tfrac{1}{2}h(y')_0 + \tfrac{1}{4}k_1\big)$$

is represented in cell D2 by the formula ....

51

=(0·5) * ($M$1^2) * ((B2 + 0·5 * $M$1) * (J2 + C2/$M$1)
+ I2 + 0·5 * $M$1 * J2+0·25 *C2)

The contents of cell D2 are then copied into cells D3 to D12.

6 Column E contains the computed k_3-values and the equation

$$k_3 = \tfrac{1}{2}h^2 f(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}h(y')_0 + \tfrac{1}{4}k_1,\ (y')_0 + k_2/h)$$
$$= \tfrac{1}{2}h^2\big((x_0 + \tfrac{1}{2}h)\big((y')_0 + k_2/h\big) + y_0 + \tfrac{1}{2}h(y')_0 + \tfrac{1}{4}k_1\big)$$

is represented in cell E2 by the formula ....

52

=(0·5) * ($M$1^2) * ((B2 + 0·5 * $M$1) * (J2 + D2/$M$1)
+ I2 + 0·5 * $M$1 * J2+0·25 *C2)

The contents of cell E2 are then copied into cells E3 to E12.

7 Column F contains the computed k_4-values and the equation

$$k_4 = \tfrac{1}{2}h^2 f(x_0 + h,\ y_0 + h(y')_0 + k_3,\ (y')_0 + 2k_3/h)$$
$$= \tfrac{1}{2}h^2\big((x_0 + h)\big((y')_0 + 2k_3/h\big) + y_0 + h(y')_0 + k_3\big)$$

is represented in cell F2 by the formula ....

53

=(0·5) * ($M$1^2) * ((B2 + $M$1) * (J2 + 2 * E2/$M$1)
+ I2 + $M$1 * J2 + E2)

The contents of cell F2 are then copied into cells F3 to F12.

8 Column G contains the computed P-values and the equation $P = \tfrac{1}{3}(k_1 + k_2 + k_3)$ is represented in cell G2 by the formula ....

54

=(1/3) * (C2+D2+E2)

The *contents* of cell G2 are then copied into cells G3 to G12.

9 Column H contains the computed Q-values and the equation $Q = \frac{1}{3}(k_1 + 2k_2 + 2k_3 + k_4)$ is represented in cell H2 by the formula ....

55

=(1/3) * (C2+2 *D2+2 * E2+F2)

The contents of cell H2 are then copied into cells H3 to H12.

10 Column I contains the computed y-values. The initial value of $y_0 = 1$ is entered into cell I2 and the equation

$$y_1 = y_0 + h(y')_0 + P$$

is represented in cell I3 by the formula ....

56

= I2 + $M$1 * J2 + G2

The contents of cell I3 are then copied into cells I4 to I12.

11 Column J contains the computed y'-values. The initial value of $(y')_0 = 0$ is entered into cell J2 and the equation $(y')_1 = (y')_0 + Q/h$ is represented in cell J3 by the formula ....

57

=J2+H2/$M$1

The contents of cell J3 are then copied into cells J4 to J12.

12 Column K contains the y''-values which are obtained from the equation $y'' = xy' + y$ which is represented in cell K2 by the formula ....

58

$$\boxed{=B2*J2+I2}$$

The contents of cell K2 are then copied into cells K3 to K12 and the final spreadsheet looks like the following

n	x	k1	k2	k3	k4	P	Q	y	y′	y″	$h = 0{\cdot}2$
0	0·0	0·0200000	0·0203000	0·0203030	0·0212182	0·0202010	0·0408081	1·0000000	0·0000000	1·0000000	
1	0·2	0·0212202	0·0227790	0·0228258	0·0251351	0·0222750	0·0458550	1·0202010	0·2040403	1·0610091	
2	0·4	0·0251322	0·0282477	0·0284035	0·0325752	0·0272612	0·0570033	1·0832841	0·4333153	1·2566102	
3	0·6	0·0325641	0·0378798	0·0382519	0·0451961	0·0362319	0·0766745	1·1972083	0·7183318	1·6282074	
4	0·8	0·0451694	0·0538673	0·0546501	0·0660061	0·0512289	0·1094035	1·3771066	1·1017045	2·2584702	
5	1·0	0·0659480	0·0801269	0·0816865	0·1003762	0·0759205	0·1633170	1·6486764	1·6487218	3·2973982	
6	1·2	0·1002542	0·1236497	0·1266912	0·1579840	0·1168650	0·2529733	2·0543413	2·4653068	5·0127095	
7	1·4	0·1577302	0·1970991	0·2030044	0·2565931	0·1859446	0·4048434	2·6642677	3·7301734	7·8865105	
8	1·6	0·2560654	0·3238945	0·3354254	0·4295622	0·3051284	0·6680891	3·5962469	5·7543906	12·8032719	
9	1·8	0·4284592	0·5483881	0·5711745	0·7411112	0·5160073	1·1362318	5·0522535	9·0948361	21·4229585	
10	2·0	0·7387844	0·9567270	1·0024949	1·3181400	0·8993354	1·9917894	7·3872279	14·7759954	36·9392186	

The errors have been dramatically reduced, as can be seen from the following table in comparison with those in Frame 47.

n	x	Exact	Error (%)
0	0·0	1·0000000	0·00
1	0·2	1·0202013	0·00
2	0·4	1·0832871	0·00
3	0·6	1·1972174	0·00
4	0·8	1·3771278	0·00
5	1·0	1·6487213	0·00
6	1·2	2·0544332	0·00
7	1·4	2·6644562	0·01
8	1·6	3·5966397	0·01
9	1·8	5·0530903	0·02
10	2·0	7·3890561	0·02

Next frame

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Now here is one for you to do entirely on your own. The method is exactly the same as before and there are no snags. Use the spreadsheet that you created for the previous example as a template for this one.

Example 2

Solve the equation

$$y'' = x - y^2$$

for $x = 0{\cdot}0(0{\cdot}2)2{\cdot}0$ where at $x = 0$, $y = 0$ and $y' = 0$.

When you have finished, check the results with the next frame

60

n	x	k1	k2	k3	k4	P	Q	y	y′	y″	h=0·2
0	0·0	0·0000000	0·0020000	0·0020000	0·0039999	0·0013333	0·0040000	0·0000000	0·0000000	0·0000000	
1	0·2	0·0040000	0·0059996	0·0059996	0·0079974	0·0053331	0·0119986	0·0013333	0·0199999	0·1999982	
2	0·4	0·0079977	0·0099915	0·0099915	0·0119731	0·0093269	0·0199789	0·0106664	0·0799930	0·3998862	
3	0·6	0·0119741	0·0139351	0·0139351	0·0158524	0·0132814	0·0278556	0·0359919	0·1798875	0·5987046	
4	0·8	0·0158546	0·0177065	0·0177065	0·0194436	0·0170892	0·0353748	0·0852508	0·3191655	0·7927323	
5	1·0	0·0194477	0·0210264	0·0210264	0·0223594	0·0205002	0·0419709	0·1661731	0·4960396	0·9723865	
6	1·2	0·0223654	0·0233782	0·0233782	0·0239421	0·0230406	0·0466068	0·2858812	0·7058940	1·1182719	
7	1·4	0·0239482	0·0239504	0·0239504	0·0232394	0·0239497	0·0476631	0·4501006	0·9389280	1·1974094	
8	1·6	0·0232395	0·0216639	0·0216639	0·0191107	0·0221891	0·0430019	0·6618359	1·1772436	1·1619732	
9	1·8	0·0190914	0·0153806	0·0153806	0·0105578	0·0166175	0·0303905	0·9194737	1·3922530	0·9545681	
10	2·0	0·0104978	0·0043750	0·0043750	−0·0026809	0·0064159	0·0084390	1·2145418	1·5442053	0·5248882	

Because

The only items that need amending from the previous spreadsheet are the references to the actual differential equation. Consequently

The formula in D2 for k_2 now reads as

= (0·5) * ($M$1^2) * (B2 + 0·5 * $M$1 − (I2 + 0·5 * $M$1 * J2 + 0·25 * C2)^2)

The formula in E2 for k_3 now reads as

= 0·5 * ($M$1^2) * (B2 + 0·5 * $M$1 − (I2 + 0·5 * $M$1 * J2 + 0·25 * C2)^2)

The formula in F2 for k_4 now reads as

= 0·5 *($M$1^2) * (B2+$M$1 − (I2 + $M$1 * J2 + E2)^2)

The formula in K2 for y'' now reads as

= B2 − I2^2

Predictor–corrector methods

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So far, all the methods that we have used for the numerical solution of differential equations have been *single-step* methods. By this is meant that, given the differential equation $y' = f(x, y)$, a set of starting values (x_0 and y_0) and a step length (h), we can then find the value of y_1. The values of x_1 and y_1 become the starting values for the next iteration and so the procedure goes on, one step at a time. More accurate methods employ a *multi-step* procedure where, instead of starting with just a single set of initial values, we use a collection of previously calculated values.

A very simple multi-step method is given by the equations

$$\bar{y}_1 = y_0 + hf(x_0, y_0)$$

$$y_1 = y_0 + \tfrac{1}{2}h\big(f(x_0, y_0) + f(x_1, \bar{y}_1)\big)$$

Here we calculate $\bar{y}_1$ first from the given initial conditions x_0 and y_0. We call this equation the *predictor* because it gives $\bar{y}_1$ as a first estimate of y_1. Using $\bar{y}_1$ in the second equation then gives a more accurate value for y_1. We call this equation the *corrector*.

An even better pair of predictor–corrector equations is given by

$$\bar{y}_{i+1} = y_i + \tfrac{1}{2}h(3f(x_i, y_i) - f(x_{i-1}, y_{i-1}))$$
$$y_{i+1} = y_i + \tfrac{1}{2}h\big(f(x_i, y_i) + f(x_{i+1}, \bar{y}_{i+1})\big) \quad \text{for } i = 0, 1, 2, 3, \ldots$$

Here, in order to use the predictor for the first time when $i = 0$ we need to know the value of $f(x_{0-1}, y_{0-1}) = f(x_{-1}, y_{-1})$, which we do not. Instead we shall use the equation $\bar{y}_1 = y_0 + hf(x_0, y_0)$ when $i = 0$.

In the next frame we shall look at an example

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Example

Solve the equation $y' = x + y$ for $x = 0{\cdot}0(0{\cdot}1)1{\cdot}0$ where $y = 1$ when $x = 0$.

We have solved this equation before in Frame 32 using the Euler–Cauchy method and have viewed the accuracy of this method when compared with the exact solution. Here we shall see that this predictor–corrector method is even more accurate. Set up the following heading on your spreadsheet

	A	B	C	D	E	F	G
1	**n**	**x**	**y***	**y**	**Exact**	**Errors (%)**	***h*=**

As usual, column A contains the iteration numbers 0 to 10 in cells A2 to A12 and column B contains the x-values stepped according to the step length $h = 0{\cdot}1$ which is in cell H1. The initial value of $y = 1$ must be entered into cell D2.

Column C contains the predictor values given by the equations

$$\bar{y}_1 = y_0 + hf(x_0, y_0)$$
$$\bar{y}_{i+1} = y_i + \tfrac{1}{2}h(3f(x_i, y_i) - f(x_{i-1}, y_{i-1})) \quad \text{for } i > 0$$

To accommodate these equations in cell C3 enter the formula ....

63

=D2 + \$H\$1 * (B2 + D2)

And in cell C4 enter the formula ....

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=D3 + 0·5 * \$H\$1 * (3 * B3 + 3 * D3 – B2 – D2)

And copy into cells C5 to C12.

Column D contains the corrector values given by the equation

$$y_{i+1} = y_i + \tfrac{1}{2}h\big(f(x_i, y_i) + f(x_{i+1}, \bar{y}_{i+1})\big)$$

To accommodate this equation in cell D3 enter the formula ....

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$$\boxed{= D2 + 0{\cdot}5 * \$H\$1 * (B2 + D2 + B3 + C3)}$$

And copy into cells D4 to D12.

We have seen that the exact solution to this equation is $2e^x - x - 1$, so this can be programmed into the sheet entering the formula

= 2 * EXP(B2) − B2 − 1 in cell E2 and then copying it into cells E3 to E12.

The final table looks as follows

n	**x**	**y***	**y**	**Exact**	**Error (%)**	$h = 0{\cdot}1$
0	0·0		1·0000000	1·0000000	0·00	
1	0·1	1·1000000	1·1100000	1·1103418	0·03	
2	0·2	1·2415000	1·2425750	1·2428055	0·02	
3	0·3	1·3984613	1·3996268	1·3997176	0·01	
4	0·4	1·5824421	1·5837303	1·5836494	−0·01	
5	0·5	1·7963085	1·7977322	1·7974425	−0·02	
6	0·6	2·0432055	2·0447791	2·0442376	−0·03	
7	0·7	2·3266093	2·3283485	2·3275054	−0·04	
8	0·8	2·6503618	2·6522840	2·6510819	−0·05	
9	0·9	3·0187092	3·0208337	3·0192062	−0·05	
10	1·0	3·4363445	3·4386926	3·4365637	−0·06	

Here the errors are significantly reduced, as seen from the comparisons below.

1	**2**	**3**
0·00	0·00	0·00
0·93	0·03	0·03
1·84	0·06	0·02
2·69	0·09	0·01
3·50	0·12	−0·01
4·25	0·14	−0·02
4·95	0·17	−0·03
5·59	0·19	−0·04
6·18	0·21	−0·05
6·73	0·23	−0·05
7·25	0·24	−0·06

Here **1** refers to Euler, **2** refers to Euler–Cauchy and **3** refers to the predictor–corrector method just used.

And that is it. There are many other more sophisticated methods for the solution of ordinary differential equations by numerical methods and a detailed study of these is a course in itself. The methods we have used give an introduction to the processes and are practical in application.

The **Revision summary** and **Can You?** checklist now follow as usual. Check them carefully and refer back to the Programme for any points that may need further brushing up. Then you will be ready for the **Test exercise**, and the **Further problems** provide further practice.

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Revision summary 9

1 *Taylor's series*

$$f(a+h)=f(a)+hf'(a)+\frac{h}{2!}f''(a)+\frac{h}{3!}f'''(a)+\dots$$

2 *Solution of first-order differential equations*
Equation $y'=f(x,y)$ with $y=y_0$ at $x=x_0$ for $x_0(h)x_n$.

(a) *Euler's method*

$$y_1=y_0+h(y')_0.$$

(b) *Euler–Cauchy method*

$$\begin{aligned}x_1&=x_0+h\\ \bar{\bar{y}}_1&=y_0+h(y')_0\\ y_1&=y_0+\tfrac{1}{2}h\{(y')_0+f(x_1,\bar{\bar{y}}_1)\}\\ (y')_1&=f(x_1,\,y_1).\end{aligned}$$

(c) *Runge–Kutta method*

$$\begin{aligned}x_1&=x_0+h\\ k_1&=hf(x_0,\,y_0)=h(y')_0\\ k_2&=hf(x_0+\tfrac{1}{2}h,\,y_0+\tfrac{1}{2}k_1)\\ k_3&=hf(x_0+\tfrac{1}{2}h,\,y_0+\tfrac{1}{2}k_2)\\ k_4&=hf(x_0+h,\,y_0+k_3)\\ \Delta y_0&=\tfrac{1}{6}(k_1+2k_2+2k_3+k_4)\\ y_1&=y_0+\Delta y_0\\ (y')_1&=f(x_1,\,y_1).\end{aligned}$$

3 *Solution of second-order differential equations*
Equation $y''=f(x,y,y')$ with $y=y_0$ and $y'=(y')_0$ at $x=x_0$ for $x=x_0(h)x_n$.

(a) *Euler's second-order method*

$$\begin{aligned}y_1&=y_0+h(y')_0+\frac{h^2}{2!}(y'')_0\\ (y')_1&=(y')_0+h(y'')_0.\end{aligned}$$

▶

(b) *Runge–Kutta method*

$$x_1 = x_0 + h$$
$$k_1 = \tfrac{1}{2}h^2 f\{x_0,\ y_0,\ (y')_0\} = \tfrac{1}{2}h^2(y'')_0$$
$$k_2 = \tfrac{1}{2}h^2 f\left\{x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}h(y')_0 + \tfrac{1}{4}k_1,\ (y')_0 + \frac{k_1}{h}\right\}$$
$$k_3 = \tfrac{1}{2}h^2 f\left\{x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}h(y')_0 + \tfrac{1}{4}k_1,\ (y')_0 + \frac{k_2}{h}\right\}$$
$$k_4 = \tfrac{1}{2}h^2 f\left\{x_0 + h,\ y_0 + h(y')_0 + k_3,\ (y')_0 + \frac{2k_3}{h}\right\}$$
$$P = \tfrac{1}{3}(k_1 + k_2 + k_3)$$
$$Q = \tfrac{1}{3}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$y_1 = y_0 + h(y')_0 + P$$
$$(y')_1 = (y')_0 + \frac{Q}{h}$$
$$(y'')_1 = f\{x_1,\ y_1,\ (y')_1\}.$$

4 *Predictor–corrector*
Equation $y' = f(x, y)$ with $y = y_0$ and $y' = (y')_0$ at $x = x_0$ for $x = x_0(h)x_n$, then

Predictor

$$\bar{y}_{i+1} = y_i + \tfrac{1}{2}h(3f(x_i,\ y_i) - f(x_{i-1},\ y_{i-1})) \quad \text{for } i = 1, 2, 3, \ldots$$
$$\bar{y}_1 = y_0 + hf(x_0,\ y_0) \quad \text{for } i = 0$$

Corrector

$$y_{i+1} = y_i + \tfrac{1}{2}h\big(f(x_i,\ y_i) + f(x_{i+1},\ \bar{y}_{i+1})\big) \quad \text{for } i = 0, 1, 2, 3, \ldots$$

Can You?

Checklist 9

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Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: **Frames**

- Derive a form of Taylor's series from Maclaurin's series and from it describe a function increment as a series of first and higher-order derivatives of the function? 1 to 3

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Describe and apply by means of a spreadsheet the Euler method, the Euler–Cauchy method and the Runge–Kutta method for first-order differential equations? 4 to 43

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Describe and apply by means of a spreadsheet the Euler second-order method and the Runge–Kutta method for second-order ordinary differential equations? 44 to 60

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Describe and apply by means of a spreadsheet a simple predictor–corrector method? 61 to 65

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 9

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1 Apply Euler's method to solve the equation

$$\frac{dy}{dx} = 1 + xy \quad \text{for} \quad x = 0(0{\cdot}1)0{\cdot}5$$

given that at $x = 0$, $y = 1$.

2 The equation $\frac{dy}{dx} = x^2 - 2y$ is subject to the initial condition $y = 0$ at $x = 1$. Use the Euler–Cauchy method to obtain function values for $x = 1{\cdot}0(0{\cdot}2)2{\cdot}0$.

3 Using the Runge–Kutta method, solve the equation

$$\frac{dy}{dx} = 1 + y - x \quad \text{for} \quad x = 0(0{\cdot}1)0{\cdot}5$$

given that $y = 1$ when $x = 0$.

▶

4 Apply Euler's second-order method to solve the equation

$y'' = y - x \quad \text{for} \quad x = 2{\cdot}0(0{\cdot}1)2{\cdot}5$

given that at $x = 2$, $y = 3$ and $y' = 0$.

5 Use the Runge–Kutta method to solve the equation

$y'' = (y'/x) + y \quad \text{for} \quad x = 1{\cdot}0(0{\cdot}1)1{\cdot}5$

given the initial conditions that at $x = 1{\cdot}0$, $y = 0$ and $y' = 1{\cdot}0$.

6 Use the predictor–corrector method in the text to solve the equation

$y' = 1 + xy \quad \text{for } x = 0(0{\cdot}1)1$

given that $x = 0$ when $y = 0$.

Further problems 9

69 Solve the following differential equations by the methods indicated.

Euler's method

1	$y' = 2x - y$	$x = 0,\ y = 1$	$x = 0(0{\cdot}2)1{\cdot}0$
2	$y' = 2x + y^2$	$x = 0,\ y = 1{\cdot}4$	$x = 0(0{\cdot}1)0{\cdot}5$

Euler–Cauchy method

3	$y' = 2 - y/x$	$x = 1,\ y = 2$	$x = 1{\cdot}0(0{\cdot}2)2{\cdot}0$
4	$y' = x^2 - 2x + y$	$x = 0,\ y = 0{\cdot}5$	$x = 0(0{\cdot}1)0{\cdot}5$
5	$y' = (y - x^2)^{\frac{1}{2}}$	$x = 0,\ y = 1$	$x = 0(0{\cdot}1)0{\cdot}5$
6	$y' = \dfrac{x + y}{xy}$	$x = 1,\ y = 1$	$x = 1{\cdot}0(0{\cdot}1)1{\cdot}5$
7	$y' = y\sin x + \cos x$	$x = 0,\ y = 0$	$x = 0(0{\cdot}1)0{\cdot}5$

Runge Kutta method

8	$y' = 2x - y$	$x = 0,\ y = 1$	$x = 0(0{\cdot}2)1{\cdot}0$
9	$y' = x - y^2$	$x = 0,\ y = 1$	$x = 0(0{\cdot}1)0{\cdot}5$
10	$y' = y^2 - xy$	$x = 0,\ y = 0{\cdot}4$	$x = 0(0{\cdot}2)1{\cdot}0$
11	$y' = \sqrt{2x + y}$	$x = 1,\ y = 2$	$x = 1{\cdot}0(0{\cdot}2)2{\cdot}0$
12	$y' = 1 - x^3/y$	$x = 0,\ y = 1$	$x = 0(0{\cdot}2)1{\cdot}0$
13	$y' = \dfrac{y - x}{y + x}$	$x = 0,\ y = 1$	$x = 0(0{\cdot}2)1{\cdot}0$

Euler second-order method

14	$y'' = (x+1)y' + y$	$x = 0,\ y = 1,\ y' = 1$	$x = 0(0{\cdot}1)0{\cdot}5$
15	$y'' = 2(xy' - 4y)$	$x = 0,\ y = 3,\ y' = 0$	$x = 0(0{\cdot}1)0{\cdot}5$

Runge–Kutta second-order method

16	$y'' = x - y - xy'$	$x = 0,\ y = 0,\ y' = 1$	$x = 0(0{\cdot}2)1{\cdot}0$
17	$y'' = (1-x)y' - y$	$x = 0,\ y = 1,\ y' = 1$	$x = 0(0{\cdot}2)1{\cdot}0$
18	$y'' = 1 + x - y^2$	$x = 0,\ y = 2,\ y' = 1$	$x = 0(0{\cdot}1)0{\cdot}5$
19	$y'' = (x+2)y - 2y'$	$x = 0,\ y = 1,\ y' = 0$	$x = 0(0{\cdot}2)1{\cdot}0$
20	$y'' = \dfrac{y - xy'}{x^2}$	$x = 1,\ y = 0,\ y' = 1$	$x = 1{\cdot}0(0{\cdot}2)2{\cdot}0$

Predictor–corrector

21	$y' = 2 - y/x$	$x = 1,\ y = 2$	$x = 1{\cdot}0(0{\cdot}2)2{\cdot}0$
22	$y' = 2x - y$	$x = 0,\ y = 1$	$x = 0{\cdot}0(0{\cdot}2)1{\cdot}0$
23	$y' = \sqrt{2x + y}$	$x = 1,\ y = 2$	$x = 1{\cdot}0(0{\cdot}2)2{\cdot}0$